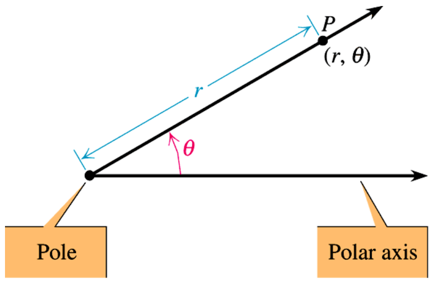
# Polar Coordinates

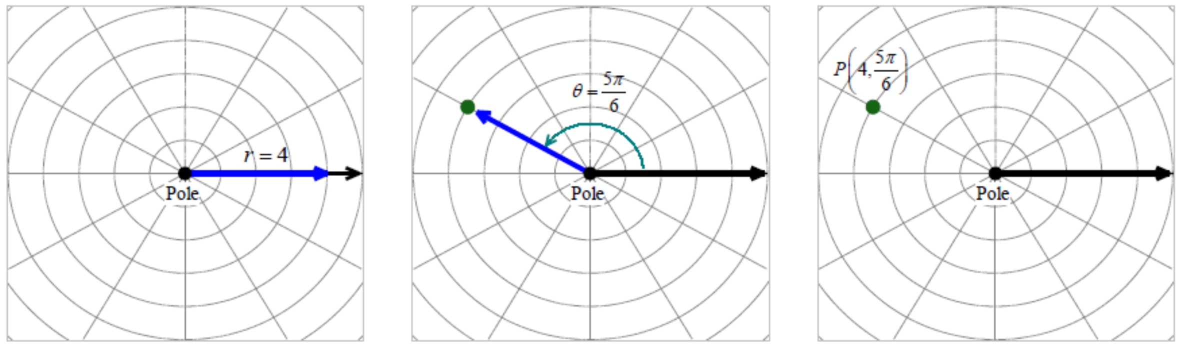



The polar coordinate system is based on a fixed point called the **pole** and a fixed axis called the **polar axis.** Points are represented by ordered pairs in the form $(r, \theta)$, where $r$ is the **directed distance** from the pole and $\theta$ is an angle whose initial side is the polar axis and whose terminal side contains the point. Typically, we choose the origin as the pole and the positive $x$-axis as the polar axis.

*To graph $(-r, \theta)$, you move in the opposite direction you would move to graph $(r, \theta)$.

**Example 8.1.1.** Plot the point $P$ with polar coordinates $\left(4, \frac{5\pi}{6}\right)$.

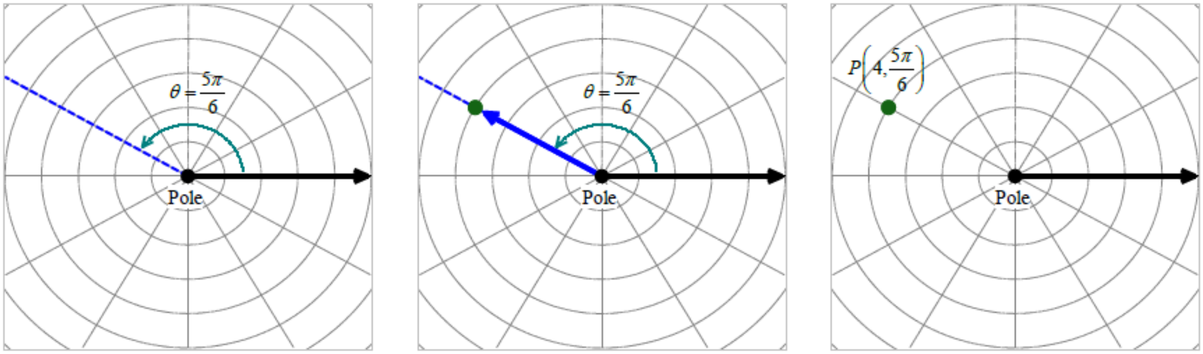
**Solution.** We start at the pole, move out along the polar axis 4 units, then rotate $\frac{5\pi}{6}$ radians counter-clockwise.




First | Second | The Resulting Point

We may also visualize this process by thinking of the rotation first.[3] To plot $P\left(4, \frac{5\pi}{6}\right)$ this way, we rotate $\frac{5\pi}{6}$ radians counter-clockwise from the polar axis, then move outwards from the pole 4 units. Essentially, we are locating a point on the terminal side of $\frac{5\pi}{6}$ which is 4 units away from the pole.

[3] As with anything in Mathematics, the more ways you have to look at something, the better. Take some time to think about both approaches to plotting points given in polar coordinates.



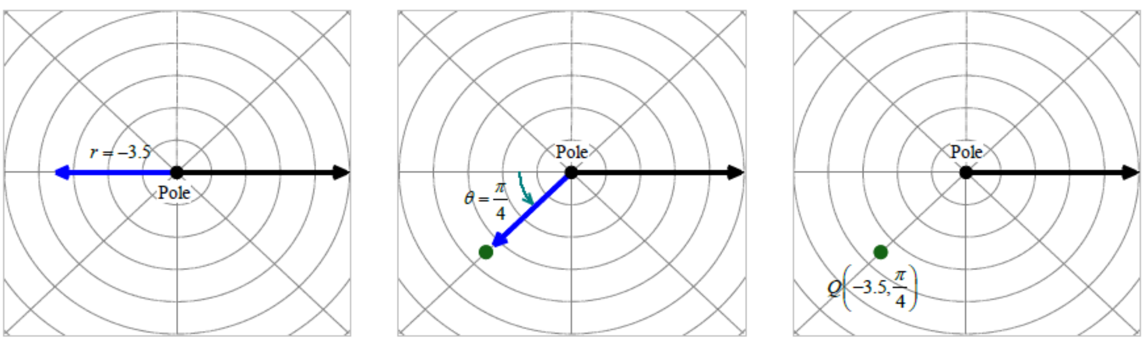

First Second The Resulting Point

□

If $r < 0$, we begin by moving, from the pole, in the opposite direction of the polar axis.

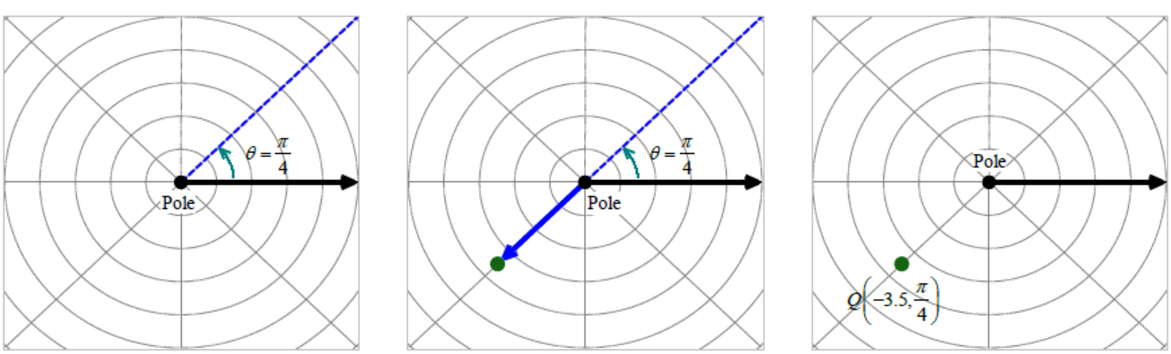
**Example 8.1.2.** Plot $Q\left(-3.5, \frac{\pi}{4}\right)$.

**Solution.** We start at the pole, moving 3.5 units in the opposite direction of the polar axis. We then rotate $\frac{\pi}{4}$ units counter-clockwise.




First Second The Resulting Point

If we interpret the angle first, we rotate $\frac{\pi}{4}$ radians, then move back through the pole 3.5 units. Here we are locating a point 3.5 units away from the pole on the terminal side of $\frac{5\pi}{4}$, not $\frac{\pi}{4}$.

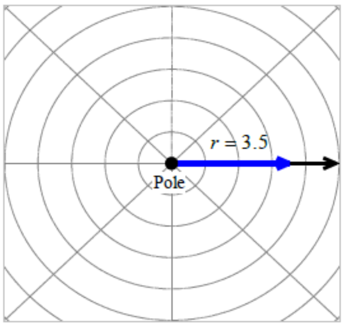



First Second The Resulting Point

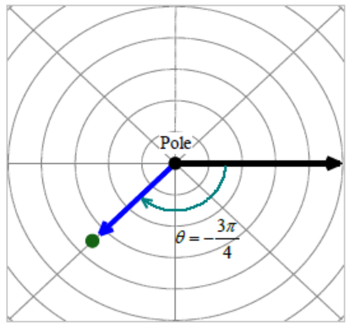
As you may have guessed, $\theta < 0$ means the rotation away from the polar axis is clockwise instead of counter-clockwise.

**Example 8.1.3.** Plot $R\left(3.5, -\frac{3\pi}{4}\right)$.

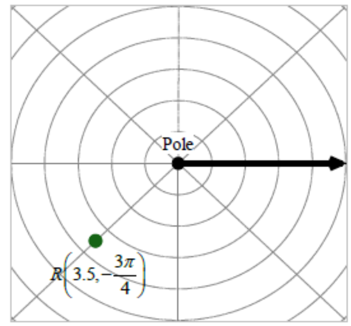
**Solution.** To plot $R\left(3.5, -\frac{3\pi}{4}\right)$, we have the following.




First

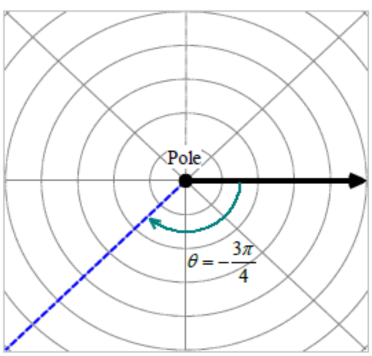



Second

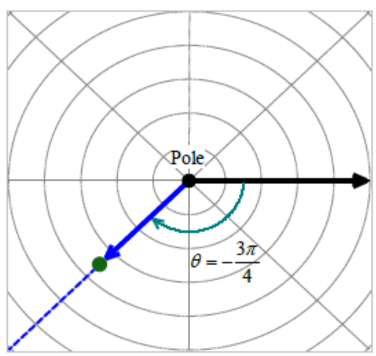



The Resulting Point

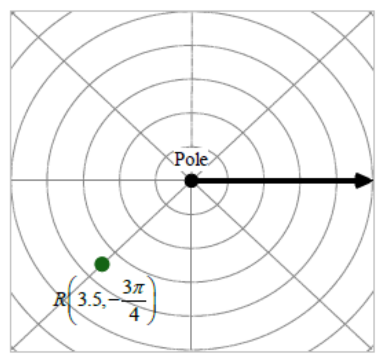
From an ‘angles first’ approach, we rotate $-\frac{3\pi}{4}$ then move out 3.5 units from the pole. We see that $R$ is the point on the terminal side of $\theta = -\frac{3\pi}{4}$ which is 3.5 units from the pole.




First



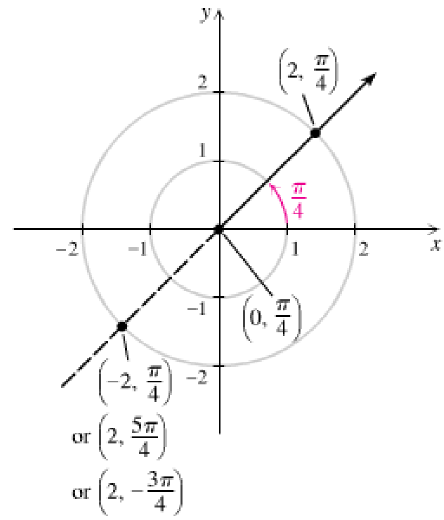

Second



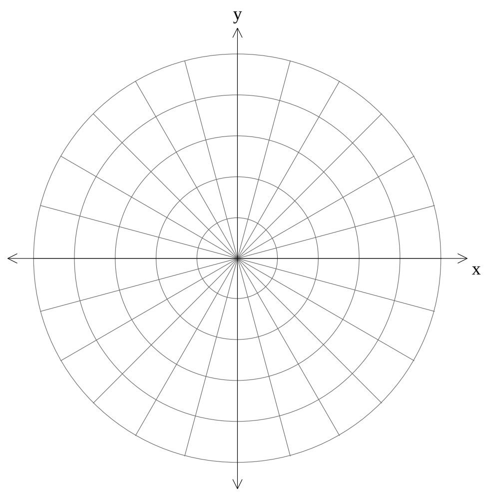

The Resulting Point

□

Polar coordinates are not unique. The points
$\left(-2,\frac{\pi}{4}\right)$, $\left(2,\frac{5\pi}{4}\right)$, and $\left(2,-\frac{3\pi}{4}\right)$ all name the same point.




**Examples:** Plot the points whose polar coordinates are given.
$A\left(3,\frac{\pi}{3}\right)$, $B\left(-1,\frac{\pi}{6}\right)$, $C\left(2,-\frac{7\pi}{4}\right)$, $D\left(-5,-\frac{3\pi}{4}\right)$, $E\left(4,\frac{\pi}{2}\right)$, $F\left(-3,\frac{2\pi}{3}\right)$




**Equivalent Representations of Points in Polar Coordinates**

Suppose $(r,\theta)$ and $(r',\theta')$ are polar coordinates where $r \neq 0$, $r' \neq 0$ and the angles are measured in radians. Then $(r,\theta)$ and $(r',\theta')$ determine the same point *P* if and only if one of the following is true:

- $r' = r$ and $\theta' = \theta + 2\pi k$ for some integer $k$
- $r' = -r$ and $\theta' = \theta + (2k+1)\pi$ for some integer $k$

All polar coordinates of the form $(0,\theta)$ represent the pole regardless of the value of $\theta$.

The key to understanding this result, and indeed the whole polar coordinate system, is to keep in mind that $(r,\theta)$ means (directed distance from pole, angle of rotation). If $r = 0$, then no matter how much rotation is performed, the point never leaves the pole. Thus, $(0,\theta)$ is the pole for all values of $\theta$.

**Polar-Rectangular Conversion Rules**

- To convert $(r,\theta)$ to rectangular coordinates $(x, y)$, use $x = r\cos\theta$ and $y = r\sin\theta$.
- To convert $(x, y)$ to polar coordinates $(r,\theta)$, use $r = \sqrt{x^2 + y^2}$ and any angle $\theta$ in standard position whose terminal side contains $(x, y)$.

**Examples:**

a) Convert $(3, 45°)$ to rectangular coordinates.

b) Convert $\left(-2, 2\sqrt{3}\right)$ to polar coordinates.